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ON MODIFIED GREEN'S FUNCTIONS IN EXTERIOR PROBLEMS FOR THE HELMHOLTZ EQUATION

by

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I Introduction

Recently, Jones [1] has presented a method for overcoming the non uniqueness problem arising in boundary integral equation formulations of the Dirichlet and Neumann problems for the Helmholtz equation. The major portion of Jones' analysis concerned the exterior Neumann problem in two dimensions although he indicated how the results generalised to three dimensions and suggested that the exterior Dirichlet problem could be similarly treated. Ursell [2] simplified the proof of a key theorem in [1] but confined his remarks to the exterior Neumann problem in two dimensions. In [3] the authors presented a systematic exposition of boundary integral equation formulations of both Dirichlet and Neumann problems and presented a number of useful properties of the boundary integral operators arising in both layer theoretic and Green's function approaches.

In particular it was shown that uniqueness of the boundary integral equation formulations of exterior problems could be retained even at eigenvalues of the corresponding adjoint interior problems by treating a pair of coupled equations. That treatment dealt with three dimensional problems although the results remain intact when the fundamental solution of the Helmholtz equation in n dimensions is used.

In the present note we show how Jones' modification can be incorporated into the boundary integral equation formalism of [3]. Ursell's simplification is adapted to three dimensions and explicit results are obtained for both Dirichlet and Neumann problems. In particular it is demonstrated that a single boundary integral equation is uniquely solvable in each case even at interior eigenvalues of the

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Recently, Jones has presented a method for overcoming the non uniqueness problem arising in boundary integral equation formulations of the Dirichlet and Neumann problems for the Helmholtz equation. The major portion of Jones' analysis concerned the exterior Neumann problem in two dimensions although he indicated how the results generalized to three dimensions and suggested that the exterior Dirichlet problem could be similarly treated. Ursell simplified the proof of a key theorem, but confined his remarks to the (CONTINUED)

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ITEM #20, CONTINUED: exterior Neumann problem in two dimensions. The authors presented a systematic exposition of boundary integral equation formulations of both Dirichlet and Neumann problems and presented a number of useful properties of the boundary integral operators arising in both layer theoretic and Green's function approaches.

In particular, it was shown that uniqueness of the boundary integral equation formulations of exterior problems could be retained even at eigenvalues of the corresponding adjoint interior problems by treating a pair of coupled equations. That treatment dealt with three dimensional problems although the results remain intact when the fundamental solution of the Helmholtz equation in n dimensions is used.

The authors show how Jones' modification can be incorporated into the boundary integral equation formalism. Ursell's simplification is adapted to three dimensions and explicit results are obtained for both Dirichlet and Neumann problems. In particular it is demonstrated that a single boundary integral equation is uniquely solvable in each case even at interior eigenvalues of the adjoint problems by suitably modifying the Green's function in a way suggested by Jones' approach. Moreover in the present note it is shown that by abandoning the restriction to real coefficients in the modification which Jones and Ursell found sufficient to eliminate non uniqueness at interior eigenvalues, the coefficients may be chosen to be optimal with respect to certain specific criteria. In particular results are presented which show how to choose the coefficients so as to minimize the difference between the modified and exact Green's functions for the Dirichlet and Neumann problems and furthermore an algorithm is presented which determines the coefficients so as to minimize the spectral radius of the modified boundary integral operator. Different coefficient choices result in each case.

An alternate treatment involving the explicit determination of a set of eigenfunctions of the unmodified operator is given.

adjoint problems by suitably modifying the Green's function in a way suggested by Jones' approach. Moreover in the present note it is shown that by abandoning the restriction to real coefficients in the modification which Jones and Ursell found sufficient to eliminate non uniqueness at interior eigenvalues, the coefficients may be chosen to be optimal with respect to certain specific criteria. In particular results are presented which show how to choose the coefficients so as to minimize the difference between the modified and exact Green's functions for the Dirichlet and Neumann problems and furthermore an algorithm is presented which determines the coefficients so as to minimize the spectral radius of the modified boundary integral operator. Different coefficient choices result in each case.

An alternate treatment involving the explicit determination of a set of eigenfunctions of the unmodified operator is given in [4].

II Notation and definitions

We adopt the notation of [3] for present purposes. Thus let D_- denote a bounded domain in $\mathbb{R}^3$ with boundary D and exterior D_+. The boundary ∂D will be assumed Lyapunov and when more stringent smoothness assumptions are needed they will be explicitly noted.

Let $R = R(P,Q)$ denote the distance between any two points P and Q in $\mathbb{R}^3$. A function $u_+(P)$ which satisfies

$$(\nabla^2 + k^2)u_+(P) = 0, \; P \in D_+ \cup \partial D, \tag{1}$$

$$\lim_{r_p \to \infty} r_p \left(\frac{\partial u_+}{\partial r_p} - iku_+\right) = 0 \tag{2}$$

will be called (following Wilcox [5]) a radiating solution of the Helmholtz equation where (r_p, θ_p, ϕ_p) are the spherical polar coordinates of a point P relative to a Cartesian coordinate system with origin in D_-. It should be stressed that the notation in (1) is meant to imply that there is an open domain including the boundary ∂D in which u_+ is differentiable. A fundamental solution of the Helmholtz equation is given by

$$\gamma_o(P,Q) := -\frac{e^{ikR}}{2\pi R} \tag{3}$$

and if g(P,Q) is a radiating solution of the Helmholtz equation in both P and Q then

$$\gamma_1(P,Q) := \gamma_o(P,Q) + g(P,Q)$$

is also a fundamental solution. If $w \in L_2(\partial D)$ then we may obtain standard and modified forms of single and double layer distributions of density w according to whether γ_o or γ_1 is employed as the fundamental solution. Thus we have, for j = 0,1,

$$(S_j w)(P) := \int_{\partial D} w(q)\ \gamma_j(P,q)ds_q, \quad P \in \mathbb{R}^3 \tag{5}$$

$$(D_j w)(P) := \int_{\partial D} w(q)\frac{\partial \gamma_j}{\partial n_q}(P,q)ds_q, \quad P \in \mathbb{R}^3 \tag{6}$$

where $\frac{\partial}{\partial n_q}$ is the derivative in the direction of the outward normal to ∂D at the point q ($\hat{n}_q$ points into D_+). Further we write $\frac{\partial}{\partial n_p^+}$ and $\frac{\partial}{\partial n_p^-}$ to denote the normal derivatives when $P \to p \in \partial D$ from D_+ and D_- respectively.

Denote by K_j the boundary integral operator, compact on $L_2(\partial D)$ as well as $C_o(\partial D)$, (Mikhlin [6]), defined by

$$(K_j w)(P) := \int_{\partial D} w(q)\ \frac{\partial \gamma_j}{\partial n_p}(P,q)\ ds_q, \ p \in \partial D \tag{7}$$

with L_2 adjoint

$$(K_j^* w)(P) := \int_{\partial D} w(q) \frac{\partial \bar{\gamma}_j}{\partial n_q}(P,q)\, ds_q, \quad p \in \partial D \tag{8}$$

where a bar is used to denote complex conjugate. In terms of K_j the jump conditions for the single and double layer distributions are

$$\frac{\partial}{\partial n_p^{\pm}}(S_j w) = \pm w + K_j w, \quad p \in \partial D \tag{9}$$

and

$$\lim_{P \to p\pm} D_j w = \mp w + \bar{K}_j^* w, \quad p \in \partial D. \tag{10}$$

Note that $g(P\ Q)$ must be defined in a neighbourhood of ∂D in rder for (9) to have meaning.

If the normal derivative of the double layer exists (e.g. if $w \in C_1(\partial D)$) then, e.g. Gunter [7],

$$\frac{\partial}{\partial n_p^+}(D_j w) = \frac{\partial}{\partial n_p^-}(D_j w) \tag{11}$$

The existence of the normal derivative of the double layer when the density arises as the solution of an integral equation, as in the present context, is established by Angell and Kleinman [8]. Král [9] discusses the case of even weaker hypothesis on the density.

If u_+ is a radiating solution of the Helmholtz equation then Green's Theorem together with (9) - (11) yield the representation

$$S_o\left(\frac{\partial u_+}{\partial n^+}\right)(P) - D_o u_+(P) = \begin{cases} 2u_+ , & P \in D_+ \\ u_+ , & P \in \partial D \\ 0 , & P \in D_- . \end{cases} \tag{12}$$

Since $g(P,q)$ must have singularities in D_- (there are no nontrivial radiating wave functions in $\mathbb{R}^3$) the representation (12) does not always hold if γ_o is replaced by γ_1. However the following is valid:

$$S_1\left(\frac{\partial u_+}{\partial n^+}\right)(P) - D_1u_+(P) = \begin{cases} 2u_+(P), & P \in D_+ \\ u_+(P), & p \in \partial D \end{cases} \qquad (13)$$

The representations (12) and 13) yield the boundary integral equations.

$$S_j\left(\frac{\partial u_+}{\partial n^+}\right) - \bar{K}_j^* u_+ = u_+, \; p \in \partial D \qquad (14)$$

$$K_j\left(\frac{\partial u_+}{\partial n^+}\right) - \frac{\partial}{\partial n}(D_j u_+) = \frac{\partial u_+}{\partial n}, \; p \in \partial D. \qquad (15)$$

Observe that γ_1 is an approximate Green's function as defined by Roach [10, 11] and the present analysis may be considered an application of the ideas presented there.

Finally we adapt Ursell's notation to $\mathbb{R}^3$ and define normalized spherical wave functions

$$C_{nm}^{e,i}(P) := \left\{-\frac{ik}{2\pi}\,\varepsilon_m(2n+1)\,\frac{(n-m)!}{(n+m)!}\right\}^{\frac{1}{2}} Z_n^{e,i}(kr_p)P_n^m(\cos\theta_p)\cos m\phi_p \qquad (16)$$

and

$$S_{nm}^{e,i}(P) := \left\{-\frac{ik}{2\pi}\,\varepsilon_m(2n+1)\,\frac{(n-m)!}{(n+m)!}\right\}^{\frac{1}{2}} Z_n^{e,i}(kr_p)P_n^m(\cos\theta_p)\sin m\phi_p$$

where

$$\varepsilon_m = \begin{cases} 1, & m = 0 \\ 2, & m > 0 \end{cases},$$

$P_n^m(\cos\theta_p)$ is the associated Legendre polynomial and $Z_n^{e,i}(kr)$ are the spherical Bessel and Hankel functions

$$Z_n^e(kr) = h_n^{(1)}(kr) \quad , \quad Z_n^i(kr) = j_n(hr).$$

Similarly we denote differentiation with respect to kr_p by a prime so that

$$C_{nm}^{e,i'}(P) := \frac{\partial}{\partial(kr_p)} C_{nm}^{e,i}(P) \quad ,$$

$$S_{nm}^{e,i'}(P) := \frac{\partial}{\partial(kr_p)} S_{nm}^{e,i}(P) \quad , \qquad (17)$$

and the differentiation only affects the radial functions.

The orthogonality of spherical harmonics leads to the relations

$$\int_{\partial B_{r_p}} \bar{C}_{nm}^{e,i'}(P) S_{\ell s}^{e,i}(P)\,ds_p = \int_{\partial B_{r_p}} C_{nm}^{e,i}(P) S_{\ell s}^{e,i'}(P)\,ds_p = 0 \;\forall n,m,\ell,s. \quad (18)$$

$$\int_{\partial B_{r_p}} C_{nm}^{e,i}(P)\bar{C}_{\ell s}^{e,i'}(P)\,ds_p = 0, \ell \neq n \text{ and/or } m \neq s$$

$$= \quad 2\,k\,r_p^2\, Z_n^{e,i}(kr_p)\, \bar{Z}_n^{e,i'}(kr_p), \qquad n = \ell,\ m = s. \quad (19)$$

and

$$\int_{\partial B_{r_p}} S_{nm}^{e,i}(P)\bar{S}_{\ell s}^{e,i'}(P)\,ds_p = 0, \ell \neq n \text{ and/or } m \neq s \text{ or } m = 0$$

$$= \quad 2\,k\,r_p^2\, Z_n^{e,i}(kr_p)\, \bar{Z}_n^{e,i'}(kr_p), \qquad n = \ell,\ m = s. \quad (20)$$

where ∂B_{r_p} is the surface of a sphere of radius $r_p = |P|$.

Because of the expansion of the fundamental solution which in this notation is

$$\gamma_o(P,Q) = \sum_{n=0}^{\infty} \sum_{n=0}^{n} \{C_{nm}^{i}(P<)C_{nm}^{e}(P>) + S_{nm}^{i}(P<)S_{nm}^{e}(P>)\} \quad (21)$$

where $P< = \begin{cases} P \text{ if } r_P < r_Q \\ Q \text{ if } r_Q < r_P \end{cases}$, $\quad P> = \begin{cases} P \text{ if } r_P > r_Q \\ Q \text{ if } r_Q > r_P \end{cases}$,

the single and double layers have the representations

$$(S_o w)(P) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \underset{\sim}{C}_{nm}^{e}\, C_{nm}^{i}(P) + \underset{\sim}{S}_{nm}^{e}\, S_{nm}^{i}(P),\ r_p \leq \min_{q\in\partial D} r_q$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{n} \underset{\sim}{C}_{nm}^{i}\, C_{nm}^{e}(P) + \underset{\sim}{S}_{nm}^{i}\, S_{nm}^{e}(P),\ r_p \geq \max_{q\in\partial D} r_q \quad (22)$$

and

$$(D_o w)(P) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \underset{\sim}{C}^{e'}_{nm} C^{i}_{nm}(P) + \underset{\sim}{S}^{e'}_{nm} S^{i}_{nm}(P), \; r_p \leq \min_{q \in \partial D} r_q$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{n} \underset{\sim}{C}^{i'}_{nm} C^{e}_{nm}(P) + \underset{\sim}{S}^{i'}_{nm} S^{e}_{nm}(P), \; r_p \geq \max_{q \in \partial D} r_q \tag{23}$$

where

$$\underset{\sim}{C}^{e,i}_{nm} := \int_{\partial D} w(q)\, C^{e,i}_{nm}(q)\, ds_q,$$

$$\underset{\sim}{C}^{e,i'}_{nm} := \int_{\partial D} w(q) \frac{\partial}{\partial n_q} C^{e,i}_{nm}(q)\, ds_q \tag{24}$$

$$\underset{\sim}{S}^{e,i}_{nm} := \int_{\partial D} q(q)\, S^{e,i}_{nm}(q)\, ds_q,$$

$$\underset{\sim}{S}^{e,i'}_{nm} := \int_{\partial D} w(q) \frac{\partial}{\partial n_q} S^{e,i}_{nm}(q)\, ds_q \tag{25}$$

III The Exterior Neumann Problem

The exterior Neumann problem consists of finding u_+, a radiating solution of the Helmholtz equation in the sense of (1) and (2) for which $\frac{\partial u_+}{\partial n} = g_+$, $p \in \partial D$ where g_+ is a given function on the boundary. Boundary integral equation formulations are obtained from Green's theorem (14) and (15).

$$(I + \bar{K}_j^*)\, \bar{w}^* = S_j\, g_+ \tag{26}$$

$$\frac{\partial}{\partial n} D_j\, \bar{w}^* = (K_j - I) g_+ \tag{27}$$

in which case the solution is represented in D_+ via (12) and (13) as

$$u_+ = \frac{1}{2} S_j g_+ - \frac{1}{2} D_j\, \bar{w}^*, \; P \in D_+. \tag{28}$$

Alternatively the layer theoretic approach starts with the assumed form of the solution

$$u_+ = (S_j w)(P), \; P \in D_+ \cup \partial D \tag{29}$$

which with (9) yields the boundary integral equation

$$(I + K_j)\, w = g_+,\ p \in \partial D \quad . \tag{30}$$

As in [3] we denote those real values of k for which the homogeneous equations

$$(I + K_j)\, \overset{\circ}{w} = 0 \tag{31}$$

has nontrivial solutions as characteristic values of $(-K_j)$ and observe, since K_j is compact on $L_2(\partial D)$, that if k is a characteristic value of $(-K_j)$ it is also a characteristic value for $(-K_j^*)$, $(-\bar{K}_j)$ and $(-\bar{K}_j^*)$. It is of course these characteristic values which impede the solution of (26) and (30). In [3] it was shown that the pair of equations (26) and (27) had a unique solution for all real values of k when $j = 0$. The fact that a pair must be considered coupled with the complicated nature of (27) makes this approach difficult to follow in actually constructing solutions, for example by numerical methods. It is this complication that is avoided by properly perturbing the fundamental solution.

First we note the following important result.

Theorem 3.1: For Im $k \geq 0$

$$(I + K_j)\, \overset{\circ}{w} = 0 \text{ if and only if } S_j\, \overset{\circ}{w} = 0,\ p \in \partial D.$$

Proof: If $\overset{\circ}{w} = 0$ the theorem holds vacuously. If $j = 0$ the result is proven in [3]. For $j = 1$ the same proof applies. $S_1 \overset{\circ}{w} = 0$, $p \in \partial D$ implies $S_1 w = 0$, $P \in D_+$ from the uniqueness of solutions of the exterior Dirichlet problem for Im $k \geqslant 0$. Hence $\frac{\partial}{\partial n_+} S_1 w = 0$ which with (9) shows that $(I + K_1)\, \overset{\circ}{w} = 0$. Conversely if $(I + K)\, \overset{\circ}{w} = 0$ then again $S_1 w = 0$, $P \in D_+ \cup \partial D$ otherwise it would violate the uniqueness theorem for the exterior Neumann problem which completes the proof.

Also we have for nontrivial $\overset{\circ}{w}$

Theorem 3.2: For $p \in \partial D$,

$(I + K_o)\,\overset{o}{w} = 0 \iff S_o\overset{o}{w} = 0 \iff k$ is an eigenvalue of the interior Dirichlet problem.

Actually $-k^2$ is the eigenvalue of the Laplacian but in the present paper, as in [3], we shall understand by eigenvalues of the interior Dirichlet (Neumann) problem those values of k for which there are non trivial solutions of $(\nabla^2 + k^2)\,u = 0$, $P \in D_-$ and $u = 0$ $(\frac{\partial u}{\partial n} = 0)$, $p \in \partial D$.

This result is proven in [3]. The fact that characteristic values of $-K_o$ are identical with eigenvalues of the interior Dirichlet problem does not generalize to K_1 because $g(P,Q)$ is not defined throughout D_-. Nevertheless, with a suitable choice of $g(P,Q)$ we may establish

Theorem 3.3: If

$$g(P,Q) = \sum_{n=0}^{\infty}\sum_{m=0}^{n}\{a_{nm}\,C^e_{nm}(P)C^e_{nm}(Q) + b_{nm}\,S^e_{nm}(P)S^e_{nm}(Q)\} \tag{32}$$

with

$$\text{either}\quad a_{nm} = 0 \quad\text{or}\quad |2a_{nm} + 1| < 1 \tag{33}$$

and

$$\text{either}\quad b_{nm} = 0 \quad\text{or}\quad |2b_{nm} + 1| < 1 \tag{34}$$

and k is a characteristic value of $-K_1$ then k is an eigenvalue of the interior Dirichlet problem.

Proof: Assume that k is a characteristic value of $-K_1$, i.e.

$$(I + K_1)\,\overset{o}{w} = 0 \tag{35}$$

has a nontrivial solution. Then Theorem 3.1 implies that $S_1\overset{o}{w} = 0$, $P \in D_+ \cup \partial D$ and since $g(P,q)$ is an analytic function of P for $r_P \neq 0$, it follows that $S_1\overset{o}{w} = 0$ is defined for $P \in D_-\setminus\{0\}$ and continuous on ∂D, Thus defining

$$u_- := S_1\overset{o}{w},\quad P \in D_-\setminus\{0\} \tag{36}$$

it follows that u_- (hence $\bar{u}_-$) is a solution of the Helmholtz equation in $D_-\setminus\{0\}$ which vanishes on ∂D. Hence

$$\int_{\partial D} \left(u_- \frac{\partial \bar{u}_-}{\partial n_p} - \bar{u}_- \frac{\partial u_-}{\partial n_p}\right) ds_p = 0. \tag{37}$$

Let r_p be the radius of a sphere B_{r_p} entirely in $D_-(r_p \leq \min_{q\in\partial D} r_q)$.

Then with Green's theorem the equation (37) may be written

$$\int_{\partial B_{r_p}} \left(u_- \frac{\partial \bar{u}_-}{\partial n_p} - \bar{u}_- \frac{\partial u_-}{\partial n_p}\right) ds_p = 0 = \int_{\partial B_{r_p}} \left(S_1\overset{o}{w} \frac{\partial}{\partial n_p} \overline{S_1\overset{o}{w}} - \overline{S_1\overset{o}{w}} \frac{\partial}{\partial n_p} S_1\overset{o}{w}\right) ds_p \tag{38}$$

No volume integral term arises because k is a characteristic value of $-K_1$, hence real. On B_{r_p}

$$S_g\overset{o}{w} = \int_{\partial D} g(P,q)\overset{o}{w}(q)\, ds_q =$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{n} \{a_{nm} \underset{\sim}{C}^e_{nm} C^e_{nm}(P) + b_{nm} \underset{\sim}{S}^e_{nm} S^e_{nm}(P)\} \tag{39}$$

where we use the notation of (24) and (25) with $\overset{o}{w}$ in place of w.

Substituting (22) and (39) in (38) we find

$$k\int_{\partial B_{r_p}} \{\sum_{n=0}^{\infty} \sum_{m=0}^{n} \left(\underset{\sim}{C}^e_{nm}\{C^i_{nm}(P) + a_{nm}C^e_{nm}(P)\} + \underset{\sim}{S}^e_{nm}\{S^i_{nm}(P) + b_{nm} S^e_{nm}(P)\}\right) \times$$

$$\times \sum_{\ell=0}^{\infty} \sum_{s=0}^{\ell} \left(\underset{\sim}{\bar{C}}^e_{\ell s} \{\bar{C}^{i'}_{\ell s}(P) + \bar{a}_{\ell s} \bar{C}^{e'}_{\ell s}(P)\} + \underset{\sim}{\bar{S}}^e_{\ell s} \{\bar{S}^{i'}_{\ell s}(P) + \bar{b}_{\ell s} \bar{S}^{e'}_{\ell s}(P)\}\right) -$$

$$- \sum_{n=0}^{\infty} \sum_{m=0}^{n} \left(\underset{\sim}{\bar{C}}^e_{nm}\{\bar{C}^i_{nm}(P) + \bar{a}_{nm} \bar{C}^e_{nm}(P)\} + \underset{\sim}{\bar{S}}^e_{nm}\{\bar{S}^i_{nm}(P) + \bar{b}_{nm} \bar{S}^e_{nm}(P)\}\right) \times$$

$$\times \sum_{\ell=0}^{\infty} \sum_{s=0}^{\ell} \left(\underset{\sim}{C}^e_{\ell s}\{C^{i'}_{\ell s}(P) + a_{\ell s} C^{e'}_{\ell s}(P)\} + \underset{\sim}{S}^e_{\ell s}\{S^{i'}_{\ell s}(P) + b_{\ell s} S^{e'}_{\ell s}(P)\}\right) \} ds_p = 0. \tag{40}$$

Upon applying (18) - (20) this becomes

$$2(kr_p)^2 \sum_{n=0}^{\infty} \sum_{m=0}^{n} \{ |\underset{\sim}{C}^e_{nm}|^2 \, (j_n j_n' + a_{nm} \, h_n^{(1)} \, j_n' + \bar{a}_{nm} \, j_n \, h_n^{(2)'} +$$

$$+ |a_{nm}|^2 \, h_n^{(1)} \, h_n^{(2)'}) \; +$$

$$+ |S^e_{nm}|^2 \, (j_n j_n' + b_{nm} \, h_n^{(1)} \, j_n' + \bar{b}_{nm} \, j_n h_n^{(2)'} + |b_{nm}|^2 h_n^{(1)} \, h_n^{(2)'}) \; -$$

$$- |C^e_{nm}|^2 \, (j_n j_n' + a_{nm} j_n h_n^{(1)'} + \bar{a}_{nm} h_n^{(2)} j_n' + |a_{nm}|^2 h_n^{(2)} h_n^{(1)'}) \; -$$

$$- |S^e_{nm}|^2 (j_n j_n' + b_{nm} j_n h_n^{(1)'} + \bar{b}_{nm} h_n^{(2)} j_n' + |b_{nm}|^2 h_n^{(2)} h_n^{(1)'}) \} = 0 \qquad (41)$$

where kr_p is the argument of all the spherical Bessel functions and the fact that $\overline{h_n^{(1)}} = h_n^{(2)}$ has been used. With the Wronskian relations

$$j_n h_n^{(2)'} - j_n' \, h_n^{(2)} = j_n' \, h_n^{(1)} - j_n \, h_n^{(1)'} = - \frac{i}{z^2} \qquad (42)$$

and

$$h_n^{(1)} \, h_n^{(2)'} - h_n^{(1)'} \, h_n^{(2)} = \frac{-2i}{z^2} \qquad (43)$$

where z is the argument of the functions concerned, we find that (41) becomes

$$- 2i \sum_{n=0}^{\infty} \sum_{m=0}^{n} \{ |\underset{\sim}{C}^e_{nm}|^2 (a_{nm} + \bar{a}_{nm} + 2|a_{nm}|^2) + |\underset{\sim}{S}^e_{nm}|^2 (b_{nm} + \bar{b}_{nm} + 2|b_{nm}|^2) \} = 0 \quad (44)$$

which may also be written as

$$\sum_{n=0}^{\infty} \sum_{m=0}^{n} \{ |\underset{\sim}{C}^e_{nm}|^2 (1 - |2a_{nm} + 1|^2) + |\underset{\sim}{S}^e_{nm}|^2 (1 - |2b_{nm} + 1|^2) \} = 0 \, . \qquad (45)$$

It then follows from the assumptions on coefficients (33) and (34) that

$$a_{nm} \, \underset{\sim}{C}^e_{nm} = b_{nm} \, \underset{\sim}{S}^e_{nm} = 0, \qquad (46)$$

hence from (39), we have $S g \overset{o}{w} = 0$, $0 < r_p \leq \min_{q \in \partial D} r_q$ but the analyticity of solutions of elliptic equations then implies that $S_g \overset{o}{w} = 0$, $P \in D_- \cup \partial D \backslash \{0\}$. Since $S_1 w^o = 0, p \in \partial D$ it follows then that $S_o \overset{o}{w} = 0, p \in \partial D$ and, with Theorem 3.2, that k is an eigenvalue of the interior Dirichlet problem.

It is noted that the sense of the inequalities in (33) and (34) could be reversed without changing the conclusion, however the sense of the inequality must be the same for all nonvanishing a_{nm} and b_{nm}. It should also be mentioned that inequalities similar to (33) and (34) were derived by Ursell [2] although he observed that by requiring Re $a_{nm} > 0$ and Re $b_{nm} > 0$ would guarantee compliance and thus attention could be restricted to real values of the coefficients. These inequalities have also been employed recently by Martin [12] who explored the relationship with the null field method.

Theorem 3.3 establishes that all characteristic values of $-K_1$ are eigenvalues of the interior Dirichlet problem. It remains to show that eigenvalues of the interior Dirichlet problem are not necessarily characteristic values of $-K_1$ with g suitably chosen. First we establish

Theorem 3.4: If g(P,Q) is as in Theorem 3.3 and $|2a_{nm}+1| < 1$, $|2b_{nm}+1| < 1$ all n,m then $-K_1$ has no characteristic values.

Proof. Assume k is a characteristic value so there exists a non-trivial $\overset{o}{w}$ which satisfies (35). The assumptions on a_{nm} and b_{nm} together with (45) imply $\underset{\sim}{C}^e_{nm} = \underset{\sim}{S}^e_{nm} = 0$ for all $n \geq 0$ and $0 \leq m \leq n$.

With (22) and (39) and the vanishing of $\underset{\sim}{C}^e_{nm}$ and $\underset{\sim}{S}^e_{nm}$ we have

$$S_1 w = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \underset{\sim}{C}^e_{nm} \{C^i_{nm}(P) + a_{nm} C^e_{nm}(P)\} +$$

$$+ \underset{\sim}{S}^e_{nm} \{S^i_{nm}(P) + b_{nm} S^e_{nm}(P)\} = 0, \ 0 < r_p < \min_{q \varepsilon \partial D} r_q. \tag{47}$$

But the analyticity of solutions of elliptic equations then implies that

$$S_1 \overset{o}{w} = 0, \ P \ \varepsilon \ D_- \cup \partial D \backslash \{0\}. \tag{48}$$

Hence

$$\frac{\partial}{\partial n} S_1 \overset{o}{w} = (-I + K_1)\overset{o}{w} = 0, \ P \ \varepsilon \ \partial D \tag{49}$$

which, with (35) implies that $\overset{o}{w} = 0$, violating the assumption, and thus establishing the result. Note that Theorem 3.4 remains valid if the sense of the inequalities satisfied by a_{nm} and b_{nm} is everywhere reversed.

We remark that the three dimensional form of Ursell's modification using the Green's function for a sphere enclosed by the scatterer [13] corresponds to a particular choice of the coefficients in Theorem 3.4. It is however not necessary to require all a_{nm} and b_{nm} to be non zero as illustrated in the following.

Theorem 3.5. If k is an eigenvalue of the interior Dirichlet problem of multiplicity ℓ then there exists a g(P,Q) with only ℓ non zero coefficients such that k is not a characteristic value of $-K_1$.

Proof. Assume the contrary so that, with Theorem 3.1, there exists a $\overset{o}{w}$ such that $S_1 w = 0$, $p \ \varepsilon \ \partial D$. Observe that $\underset{\sim}{S}^e_{nm}$ and $\underset{\sim}{C}^e_{nm}$ cannot vanish for all n and m otherwise, as in the proof of Theorem 3.4, $S_1 w = 0$ everywhere causing w to vanish contrary to assumption. However, as shown in the proof of theorem 3.3, in order for k to be a characteristic value of $-K_1$ it follows from (45) that

$|\underset{\sim}{C}^{e}_{nm}|^2(|2a_{nm}+1|^2-1) = |\underset{\sim}{S}^{e}_{nm}|^2(|2b_{nm}+1|^2-1) = 0$ for all n and m. We choose only one a_{nm} or b_{nm} to lie in the complex plane inside the circle of radius $\frac{1}{2}$ and center at $-\frac{1}{2}$ when $\underset{\sim}{C}^{e}_{nm}$ or $\underset{\sim}{S}^{e}_{nm}$ is non zero to have a contradiction. If k is an eigenvalue of multiplicity ℓ then the above argument may be repeated ℓ times leading to at most ℓ non zero coefficients. With such a g, k cannot be a characteristic value of $-K_1$. Additional coefficients may also be taken to be non zero without disturbing the result. By repetition of this argument there follows

Theorem 3.6. If $k_1, k_2, \ldots, k_N$ are eigenvalues (not necessarily ordered) of the interior Dirichlet problem of multiplicity $\ell_1, \ell_2, \ldots, \ell_N$ then there exists a g with $\sum_{i=1}^{N} \ell_i$ non zero coefficients such that $k_1, k_2, \ldots, k_N$ are not characteristic values of $-K_1$, hence also not characteristic values of $-\bar{K}_1$, $-K^*_1$ and $-\bar{K}_1{}^*$.

IV The Exterior Dirichlet Problem

The exterior Dirichlet problem consists of finding v_+, a radiating solution of the Helmholtz equation in the sense of (1) and (2) for which $v_+ = f_+$, $p \in \partial D$ where f_+ is given in ∂D. Boundary integral equations are obtained from Green's theorem, (14) and (15) which become in this case

$$S_j w = (I + \bar{K}_j^*)\, f_+, \; p \in \partial D \tag{50}$$

and

$$(I - K_j)w = -\frac{\partial}{\partial n} D_j\, f_+ \tag{51}$$

in which case the solution is represented in D_+ via (12) and (13) as

$$v_+ = \frac{1}{2} S_j w - \frac{1}{2} D_j f_+, \; P \in D_+. \tag{52}$$

Implicit in this formulation is the existence of the normal derivative of the double layer. This can be assured in (51) if f_+ is differentiable on ∂D and this can be relaxed somewhat by requiring the boundary to be Lipschitz (Lyapunov of order 1) [7]. In the uniqueness theorem cited below however there is always an implicit assumption that the double layer distribution has a normal derivative in a sense sufficient to allow use of Green's theorem. That this assumption is justified is proved in [8].

Alternatively the layer approach starts with the assumed form of the solution

$$v_+ = -D_j \bar{w}^*, \quad P \in D_+ \tag{53}$$

which with (10) yields the boundary integral equation

$$(I - \bar{K}_j^*)\, \bar{w}^* = f_+. \quad p \in \partial D. \tag{54}$$

As before we denote real values of k for which

$$(I - K_j)\overset{o}{w} = 0 \tag{55}$$

has nontrivial solutions as characteristic values of K_j which are also characteristic values of K_j^*, $\bar{K}_j$, and $\bar{K}_j^*$.

Paralleling the discussion in [3] for K_o we first note

Theorem 4.1. $(I-\bar{K}_j^*)\overset{o}{\bar{w}} = 0$ if and only if $\frac{\partial}{\partial n} D_j \overset{o}{\bar{w}}{}^* = 0$, $p \in \partial D$.

Proof. For $j = 0$ the proof is given in [3]. For $j = 1$ the argument is identical. Assume $v_+ = -D_1 \overset{o}{\bar{w}}{}^*$ where $(I - \bar{K}_j^*)\overset{o}{\bar{w}} = 0$. Then (10) implies $v_+ = 0$ on ∂D and, because there are no eigenvalues of the exterior Dirichlet problem, $v_+ = 0$ in $D \cup \partial D$. Hence $\frac{\partial}{\partial n^+}(D_+ \overset{o}{\bar{w}}{}^*)$ exists and in fact vanishes. Since $\overset{o}{\bar{w}}$ is an eigenfunction of a weakly singular integral equation it is continuous on ∂D ([6]) and this fact together with the assumption that ∂D is Lyapunov of order 1 ensures that $\frac{\partial}{\partial n_-}(D_1 \overset{\circ}{\bar{w}}{}^*)$ exists and (11) holds ([7]), which establishes the desired result. Conversely if $\frac{\partial}{\partial n}(D_1 \overset{\circ}{\bar{w}}{}^*) = 0$ on ∂D let $v_+ = D_1 \overset{\circ}{\bar{w}}{}^*$ which then must vanish because of the absence of eigenvalues of the exterior Neumann problem. Since v_+ then vanishes on ∂D, use of (10) completes the proof.

Also, as shown in [3] we have

Theorem 4.2

$$(I-\bar{K}_o^*)\overset{\circ}{w}{}^* = 0 \iff \frac{\partial}{\partial n}(D_o \overset{\circ}{w}{}^*) = 0 \iff k \text{ is an eigenvalue of the}$$

interior Neumann problem.

As in Section III, the fact that characteristic values of K_o are identical with eigenvalues of the interior Neumann problem does not generalise to K_1 since $g(P,Q)$ is not defined throughout D_-. Nevertheless we can establish the following.

Theorem 4.3. If g is defined by (32) subject to (33) and (34) and k is a characteristic value of K_1 then k is an eigenvalue of the interior Neumann problem.

Proof: Assume $(I-K_1)\overset{\circ}{w} = 0$ has a nontrivial solution which means that $(I-\bar{K}_1^*)\overset{\circ}{w} = 0$ also has a nontrivial solution. Theorem 4.1 then implies that $\frac{\partial}{\partial n}(D_1 \overset{\circ}{w}{}^*) = 0$.

Define

$$v_-(P) := (-D_1 \overset{\circ}{\bar{w}}{}^*)(P) = -\int_{\partial D} \overset{\circ}{\bar{w}}{}^*(q) \frac{\partial}{\partial n_q}\{\gamma_o(P,q) + g(P,q)\} ds_q, \quad P \in D_- \setminus \{0\}. \tag{56}$$

and with (23) and (32)

$$v_-(P) = -\sum_{n=0}^{\infty}\sum_{m=0}^{n} \mathcal{C}^{e'}_{nm}\{C^i_{nm}(P) + a_{nm} C^e_{nm}(P)\} +$$

$$+ \mathcal{S}^{e'}_{nm}\{S^i_{nm}(P) + b_{nm} S^i_{nm}(P)\}, \; r_p < \min_{q \in \partial D} r_q \tag{57}$$

where $\underset{\sim}{C}^{e'}_{nm}$ and $\underset{\sim}{S}^{e'}_{nm}$ are defined in (24) and (25) with $\overset{\circ}{w}{}^*$ replacing w_o.
Since $\frac{\partial v_-}{\partial n^-} = 0$, hence $\frac{\partial \bar{v}_-}{\partial n^-} = 0$ on D, it follows that, with Green's theorem

$$\int_{\partial D} \left(v_- \frac{\partial \bar{v}_-}{\partial n} - \bar{v}_- \frac{\partial v_-}{\partial n}\right) ds_p = 0 = \int_{\partial B_r} \left(v_- \frac{\partial \bar{v}}{\partial n} - \bar{v}_- \frac{\partial v_-}{\partial n}\right) ds_p . \qquad (58)$$

Now the analysis proceeds exactly as in the proof of Theorem 3.3 with $\underset{\sim}{C}^{e'}_{nm}$ and $\underset{\sim}{S}^{e'}_{nm}$ replacing $\underset{\sim}{C}^{e}_{nm}$ and $\underset{\sim}{S}^{e}_{nm}$ respectively leading to the conclusion that

$$a_{nm}\underset{\sim}{C}^{e'}_{nm} = b_{nm}\underset{\sim}{S}^{e'}_{nm} = 0 \text{ for all } n \geq 0 \text{ and } 0 \leq m \leq n. \qquad (59)$$

This in turn implies that

$$(D_g \overset{\circ}{w}{}^*)(P) := \int_{\partial D} \overset{\circ}{w}{}^*(q) \frac{\partial}{\partial n_q} g(P,q)\, ds_q = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \{a_{nm}\underset{\sim}{C}^{e'}_{nm} C^{e}_{nm}(P) + b_{nm}\underset{\sim}{S}^{e'}_{nm} S^{e}_{nm}(P)\} \qquad (60)$$

vanishes identically which means that

$$D_1 \overset{\circ}{w}{}^* = D_o \overset{\circ}{w}{}^* \text{ for all } P. \qquad (61)$$

But $\frac{\partial}{\partial n} D_1 \overset{\circ}{w}{}^* = 0$ hence $\frac{\partial}{\partial n} D_o \overset{\circ}{w}{}^* = 0$ and Theorem 4.2 then ensures that k is an eigenvalue of the interior Neumann problem, thus completing the proof. Note that the theorem remains valid if the inequalities in (33) and (34) are reversed.

While all characteristic values of K_1 are eigenvalues of the interior Neumann problem, the converse is not necessarily true as is evident from the following:

Theorem 4.4

If g is given by (3.2) and $|2a_{nm} + 1| < 1$, $|2b_{nm}+1| < 1$ for all n,m then K_1 has no characteristic values.

Proof: Assume k is a characteristic value so that there exists a nontrivial $\overset{\circ}{w}{}^*$ such that $(I-\bar{K}_1^*)\ \overset{\circ}{w}{}^* = 0$. The fact that a_{nm} and b_{nm} are all non zero, together with (59) implies that $C^{e'}_{nm} = S^{e'}_{nm} = 0$ for all $n \geq 0$ and $0 \leq m \leq n$ Hence with (23) and (60) it is evident that

$$D_1\ \overset{\circ}{w}{}^* = 0,\ 0 < r_p \leq \min_{q\varepsilon\partial D} r_q. \tag{62}$$

But the analyticity of solutions of elliptic equations then ensures that $D_1\ \overset{\circ}{w}{}^* = 0$, $P \varepsilon D_- \setminus \{0\}$ from which it follows that

$$(I + \bar{K}^*)\ \overset{\circ}{w}{}^* = 0 \tag{63}$$

and

$$\frac{\partial}{\partial n} D_1\ \overset{\circ}{w}{}^* = 0. \tag{64}$$

But (64) together with Theorem 4.1 imply that $(I-\bar{K}^*)\ \overset{\circ}{w}{}^* = 0$ which, with (63), guarantees that $w = 0$ thus violating the original assumption and establishing the theorem. As in the exterior Neumann problem useful results obtain even with only a finite number of non zero coefficients in the representation of g.

Theorem 4.5.

If k is an eigenvalue of the interior Neumann problem of multiplicity ℓ, then there exists a $g(P,Q)$ with only ℓ non zero coefficients such that k is not a characteristic value of K_1.

Proof: Assume that k is a characteristic value of K_1. Proceeding as in the proof of Theorem 4.3 it follows that $a_{nm}\underset{\sim}{C}^{e'}_{nm} = b_{nm}\underset{\sim}{S}^{e'}_{nm} = 0$, as in (59). But $\underset{\sim}{C}^{e'}_{nm}$ and $\underset{\sim}{S}^{e'}_{nm}$ cannot vanish for all $n \geq 0$, $0 \leq m \leq n$ or else as in the proof of Theorem 4.4, $\overset{\circ}{w}{}^*$ would vanish identically. By choosing a_{nm} or b_{nm} different from zero for one case when $\underset{\sim}{C}^{e'}_{nm}$ or $\underset{\sim}{S}^{e'}_{nm}$ is unequal to zero contradicts (59) showing that k is not a characteristic value of K. If k is an eigenvalue of multiplicity ℓ

then the argument may be repeated ℓ times with linearly independent $w_1, w_2, \ldots, w$ leading to at most ℓ non zero coefficients. Repetition of this argument for different eigenvalues establishes

Theorem 4.6.

If $k_1, \ldots, k_N$ are eigenvalues (not necessarily ordered) of the interior Neumann problem of multiplicity $\ell_1, \ell_2, \ldots, \ell_N$ respectively then there exists a g with $\sum_{i=1}^{N} \ell_i$ non zero coefficients such that $k_1, \ldots, k_N$ are not characteristic values of K_1, hence also not characteristic values of $\bar{K}_1$, K_1^*, and $\bar{K}^*$.

V Optimal Modifications

Thus far we have shown that by modifying the Green's function by adding a radiating term of the form given by (32) with coefficients subject only to the inequalities (33) and (34), the characteristic values of $\pm K_1$ may be removed. Now we consider the question of choosing the coefficients more specifically so as to satisfy various desirable criteria. One obvious criterion is to choose the modified Green's function to be the exact Green's function for the problem if there exists a coefficient choice which will accomplish this, that is, if the functions $\{C^e_{nm}(P)C^e_{nm}(Q), S^e_{nm}(P)S^e_{nm}(Q)\}$ are complete in $D_+ \otimes D_+$.

The task of finding such coefficients, which is equivalent to determining the Green's function for the problem is clearly formidable. However by modifying the requirement somewhat we arrive at a coefficient choice which is optimal in one sense of approximating the Green's function.

Consider the Dirichlet problem first and denote the modified Green's function in this case by $\gamma_1^D(P,q)$. Note that if γ_1^D were the exact Green's function it would vanish for all $q \in \partial D$ and $P \in D_+$. In particular it would vanish on any sphere enclosing ∂D. Thus

Theorem 5.1: The choice of coefficients

$$a_{nm} = - \frac{(C^i_{nm}, C^e_{nm})_{\partial D}}{\|C^e_{nm}\|^2_{L_2(\partial D)}} \quad \text{and} \quad b_{nm} = - \frac{(S^i_{nm}, S^e_{nm})_{\partial D}}{\|S^e_{nm}\|^2_{L_2(\partial D)}} \tag{65}$$

minimizes the quantity

$$\int_{r_p=A} \|\gamma^D_1\|^2_{L_2(\partial D)}\, ds_p \quad \text{for all } A \geq \max_{q\in\partial D} r_q$$

where

$$(u,v)_{\partial D} := \int_{\partial D} u(q)\bar{v}(q)\,ds_q$$

and

$$\|\cdot\|_{L_2(\partial D)} := \sqrt{(.,.)} \quad .$$

This choice of coefficients minimizes the difference between the modified Green's function and the exact Green's function on $L_2\,(\partial D) \times L_2\,(\partial B_A)$ for any $A \geq \max_{q\in\partial D} r_q$.

Proof: Using the definitions (21) and (32) it follows that for $A \geq \max_{q\in\partial D} r_q$

$$\int_{r_p=A} \|\gamma^D_1\|^2_{L_2(\partial D)}\, ds_p = \int_{r_p=A} \int_{\partial D} \Big| \sum_{n=0}^{\infty} \sum_{m=0}^{n} \{C^e_{nm}(P)\,[C^i_{nm}(q) + a_{nm}C^e_{nm}(q)] +$$

$$+ S^e_{nm}(P)\,[S^i_{nm}(q) + b_{nm}S^e_{nm}(q)]\} \Big|^2 ds_q\, ds_p . \tag{66}$$

With the orthogonality of C^e_{nm} (P) and S^e_{nm} (P) on $r_p = A$ this becomes

$$\int_{r_p=A} ||\gamma_1^D||^2_{L(\partial D)} ds_p = \sum_{n=0}^{\infty} \sum_{m=0}^{n} 2kA^2 \; |h_n^{(1)}(ka)|^2 \int_{\partial D} \{$$

$$\{|C^i_{nm}(q)+a_{nm}C^e_{nm}(q)|^2+|S^i_{nm}(q)+b_{nm}S^e_{nm}(q)|^2\}ds_q$$

$$= 2kA^2 \sum_{n=0}^{\infty} \sum_{m=0}^{n} |h_n^{(1)}(kA)|^2\{||C^i_{nm}||_{L_2(\partial D)}+a_{nm}(C^e_{nm},C^i_{nm})_{\partial D}+\overline{a_{nm}}(C^i_{nm},C^e_{nm})_{\partial D} +$$

$$+|a_{nm}|^2||C^e_{nm}||^2_{L_2(\partial D)}+||S^i_{nm}||^2_{L_2(\partial D)}+b_{nm}(S^e_{nm},S^i_{nm})_{\partial D}+\bar{b}_{nm}(S^i_{nm},S^e_{nm})_{\partial D} +$$

$$+ |b_{nm}|^2||S^e_{nm}||_{L_2(\partial D)}\} \tag{67}$$

which may also be written

$$\int_{r_p=A} ||\gamma_1^D||_{L_2(\partial D)} ds_p = 2kA^2 \sum_{n=0}^{\infty} \sum_{m=0}^{n} |h_n^{(1)}(kA)|^2\{||C^i_{nm}||^2_{L_2(\partial D)} - \frac{|(C^i_{nm},C^e_{nm})_{\partial D}|^2}{||C^e_{nm}||^2_{L_2(\partial D)}} +$$

$$+||C^e_{nm}||^2_{L_2(\partial D)}\left|a_{nm}+\frac{(C^i_{nm},C^e_{nm})_{\partial D}}{||C^e_{nm}||^2_{L_2(\partial D)}}\right|^2+||S^i_{nm}||^2_{L_2(\partial D)}-\frac{|(S^i_{nm},S^e_{nm})_{\partial D}|^2}{||S^e_{nm}||^2_{L_2(\partial D)}} +$$

$$+||S^e_{nm}||^2_{L_2(\partial D)}\left|b_{nm}+\frac{(S^i_{nm},S^e_{nm})_{\partial D}}{||S^e_{nm}||^2_{L_2(\partial D)}}\right|^2 \tag{68}$$

and this is clearly minimized when a_{nm} and b_{nm} are chosen to satisfy (65), thus establishing the theorem.

A natural question is whether the choice of coefficients in Theorem 5.1 satisfies conditions (33) and (34) ensuring that K_1 has no characteristic values. This is in fact the case which we state as Theorem 5.2.

If k is real, ∂D is not a sphere, and a_{nm} and b_{nm} are chosen as in Theorem 5.1 then $|2a_{nm}+1|<1$ and $|2b_{nm}+1|<1$.

Proof: With a_{nm} defined by (65) we have

$$|2a_{nm}+1| = \left|-\frac{2(C^i_{nm},C^e_{nm})_{\partial D}}{\|C^e_{nm}\|^2_{L_2(\partial D)}}+1\right| = \left|\frac{(C^e_{nm},C^e_{nm})_{\partial D}-2(C^i_{nm},C^e_{nm})_{\partial D}}{\|C^e_{nm}\|^2_{L_2(\partial D)}}\right| . \tag{69}$$

But if k is real,

$$C^e_{nm}(q) - 2C^i_{nm}(q) = i\,\bar{C}^e_{nm}(q) \tag{70}$$

hence

$$|2a_{nm}+1| = \frac{|(\bar{C}^e_{nm},C^e_{nm})_{\partial D}|}{\|C^e_{nm}\|^2_{L_2(\partial D)}} . \tag{71}$$

or, denoting by u and v the real and imaginary part of $C^e_{nm}(q)$

$$|2a_{nm}+1| = \left|\frac{(u,u)_{\partial D}-(v,v)_{\partial D}-2i(u,v)_{\partial D}}{(u,u)_{\partial D}+(v,v)_{\partial D}}\right|$$

$$= \frac{\sqrt{((u,u)+(v,v))^2+4\{(u,v)^2-(u,u)(v,v)\}}}{(u,u)+(v,v)} \tag{72}$$

and this is clearly less than one if

$$(u,v)^2 < (u,u)(v,v) \tag{73}$$

i.e. if strict inequality holds in the Schwartz inequality. However this will be the case provided u and v are linearly independent or explicitly if

$$= j_n(kr_q)P^m_n(\cos\theta_q)\cos m\phi_q \tag{74}$$

and

$$= n_n(kr_q)P^m_n(\cos\theta_q)\cos m\phi_q \tag{75}$$

are linearly independent on ∂D. But j_n and n_n are linearly independent solutions of the spherical Bessel equation, provided r_q varies on ∂D

which it will as long as ∂D is not a sphere. This establishes the desired inequality for a_{nm} and precisely the same procedure establishes the same result for b_{nm}. When ∂D is a sphere the inequalities (33) and (34) are not satisfied. However in that case explicit results are available to show that again K_1 has no characteristic values. This is demonstrated in Section VI.

A similar result is available if we require the modified Green's function, now denoted by γ_1^N (P,q) to approximate the Green's function for the Neumann problem. Specifically we have

Theorem 5.3

The choice of coefficients

$$a_{nm} = -\frac{\left(\frac{\partial}{\partial n_q} C^i_{nm}, \frac{\partial}{\partial n_q} C^e_{nm}\right)_{\partial D}}{\left\| \frac{\partial}{\partial n_q} C^e_{nm} \right\|_{L_2(\partial D)}} \quad \text{and} \quad b_{nm} = -\frac{\left(\frac{\partial}{\partial n_q} S^i_{nm} \; \frac{\partial}{\partial n_q} S^e_{nm}\right)_{\partial D}}{\left\| \frac{\partial}{\partial n_q} S^e_{nm} \right\|_{L_2(\partial D)}} \qquad (76)$$

minimizes the quantity

$$\int_{r_p=A} \left\| \frac{\partial \gamma_1^N}{\partial n_q} \right\|^2_{L_2(\partial D)} ds_p \quad \text{for every } A \geq \max_{q\varepsilon\partial D} r_q .$$

The proof of this theorem is precisely the same as the proof of Theorem 5.1 with $C^e_{nm}(q)$ and $S^e_{nm}(q)$ replaced by $\frac{\partial}{\partial nq} C^e_{nm}(q)$ and $\frac{\partial}{\partial nq} S^e_{nm}(q)$ and similarly for C^i_{nm} and S^i_{nm}. The question again arises as to whether the coefficients specified in (76) satisfy conditions (33) and (34) guaranteeing that $-K_1$ has no characteristic values. A procedure exactly analogous to the proof of theorem 5.2, using the definitions of a_{nm} and b_{nm} given in theorem 5.3 shows that the inequality $|2a_{nm} + 1| < 1$ is satisfied provided that

$\frac{\partial}{\partial n_q}\{j_n(kr_q)P_n^m(\cos\theta_q)\cos m\phi_q\}$ and $\frac{\partial}{\partial n_q}\{n_n(kr_q)P_n^m(\cos\theta_q)\cos m\phi_q\}$ are linearly independent on ∂D. While this appears reasonable when ∂D is not a sphere, especially in view of the comparable result in Theorem 5.2 involving the undifferentiated functions, a rigorous proof has thus far eluded the authors. Once established, a similar procedure would show that the inequality $|2b_{nm}+1| < 1$ is also satisfied.

Yet another criterion for choosing the coefficients in the modification, and for some purposes perhaps the most meaningful criterion is to choose the coefficients so as to minimize a bound on the spectral radius of the modified boundary integral operator, K_1. This consists of minimizing $||K_1||$ and is equally applicable in both Dirichlet and Neumann problems. An algorithm which accomplishes this may also be derived. For this analysis, however, it is convenient to change notation slightly since the representation of the modification as as double sum (32) is somewhat awkward for this purpose. Note that each pair of integers (n,m), $0\le m\le n$ uniquely determines an integer ℓ by

$$\ell := \frac{n^2+n}{2} + m, \quad 0 \le m \le n \tag{77}$$

and moreover each $\ell \geq 0$ uniquely determines a pair (n,m) by the same relation, since ℓ lies between the sum of the first n and the first $n+1$ integers for some n. With this relationship between the integers ℓ and the ordered pairs (n,m) any series of the form $\sum_{n=0}^{\infty}\sum_{m=0}^{n} A_{nm}$ may be rewritten as $\sum_{\ell=0}^{\infty} A_\ell$. Moreover by defining

$$v_{2\ell}(P) := C_\ell^e(P) = C_{nm}^e(P) \;; \quad v_{2\ell+1}(P) := S_\ell^e(P) = S_{nm}^e(P) \tag{78}$$

$$\alpha_{2\ell} := a_\ell = a_{nm} \quad ; \quad \alpha_{2\ell+1} := b_\ell = b_{nm} \tag{79}$$

We may rewrite the modification (32) as

$$g(P,q) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \{a_{nm} C^e_{nm}(P) C^e_{nm}(q) + b_{nm} S^e_{nm}(P) S^e_{nm}(q)\} = \sum_{\ell=0}^{\infty} \alpha_\ell \, v_\ell(P) v_\ell(q) \tag{80}$$

It is also useful for what follows to have the following

Lemma 1: The sets $\{v_\ell\}_{\ell=0}^N$ and $\{\frac{\partial v_\ell}{\partial n}\}_{\ell=0}^N$ are linearly independent on ∂D for any N including $(+\infty)$.

Proof. Consider first the set $\{v_\ell\}_{\ell=0}^N$ where v_ℓ is defined in (78). Assume linear dependence i.e. there exist $\{C_\ell\}_{\ell=0}^N$ not all zero such that

$$\phi(P) := \sum_{\ell=0}^{N} C_\ell v_\ell(P) = 0, \quad P \in \partial D.$$

But $\phi(P)$, $P \in D_+$ is a radiating solution of the Helmholtz equation with zero boundary values on ∂D. Therefore the uniqueness of solutions of the exterior Dirichlet problem implies that $\phi(P) = 0$, $P \in D_+$. In particular $\phi(P) = \sum_{\ell=0}^{N} C_\ell v_\ell(P) = 0$ on any sphere containing ∂D. But $\{v_\ell\}$ are known to be linearly independent on spheres hence $C_\ell = 0 \ \forall \ell$ violating our assumption and thus establishing the linear independence of $\{v_\ell\}$ on ∂D. Similarly the uniqueness of solutions of the exterior Neumann problem implies that if

$$\psi(P) := \sum_{\ell=0}^{N} C_\ell v_\ell(P), \ P \in D_+ \ \text{ and}$$

$$\frac{\partial \psi}{\partial n} = \sum_{\ell=0}^{N} C_\ell \frac{\partial v_\ell}{\partial n} = 0, \quad P \in \partial D$$

then $\psi \equiv 0$, $P \in D_+$ and again the linear independence on spheres implies the linear independence of $\{\frac{\partial v_\ell}{\partial n}\}$ on ∂D.

Since $\{v_\ell\}_{\ell=0}^{N}$ are linearly independent on ∂D, though not orthogonal, there does exist a dual basis of the span of $\{v_\ell\}$ denoted by $\{v_\ell^\perp\}_{\ell=0}^{N}$ with the property that

$$(v_\ell,\ v_m^\perp)_{\partial D} = \delta_{\ell m}. \tag{81}$$

In fact the functions $v_m^\perp$ may be represented in terms of $\{v_\ell\}$ by

$$v_m^\perp(P) = \sum_{j=0}^{N} C_{mj}\ v_j\ (P) \tag{82}$$

where the coefficients C_{mj} are solutions of the equations

$$\sum_{j=0}^{N} \bar{C}_{mj}\ (v_\ell, v_j)_{\partial D} = \sum_{j=0}^{N} C_{mj}\ (v_j, v_\ell)_{\partial D} = \delta_{\ell m}. \tag{83}$$

For each m the set of N+1 equations is uniquely solvable because the linear independence of $\{v_\ell\}$ implies that the coefficient matrix with elements $(v_\ell, v_j)_{\partial D}$ is non singular.

We now are able to characterize the coefficients of the modified Green's function which minimize the spectral radius of K_1.

Theorem 5.4

The choice of coefficients

$$\alpha_\ell = \frac{-\int_{\partial D}\int_{\partial D}\frac{\partial}{\partial n_p}\gamma_o(P,q)\ \frac{\partial}{\partial n_p}\ \bar{v}_\ell(P)\bar{v}_\ell^\perp(q)\,ds_p ds_q}{\left\|\ \frac{\partial v_\ell}{\partial n_p}\ \right\|^2_{L^2(\partial D)}} \tag{84}$$

in the modification (80) minimizes $\|K_1\|_{L_2(D)}$ hence minimizes an upper bound on the spectral radius of K_1.

Proof: The operator norm will be minimized if the coefficients in the modification minimize $\|K_1 w\|^2_{L_2(\partial D)}$ for each $w \in L_2(\partial D)$. But in the present notation

$$||K_1 w||^2_{L_2(\partial D)} = (K_1 w, K_1 w)_{\partial D} = \int_{\partial D}\int_{\partial D}\int_{\partial D} \frac{\partial \gamma_1}{\partial n_p}(P,q)\, \frac{\partial \bar{\gamma}_1}{\partial n_p}(P,q_1) w(q)\bar{w}(q_1)\, ds_q\, ds_{q_1}\, ds_p =$$

$$= \int_{\partial D}\int_{\partial D}\int_{\partial D} \Big[\frac{\partial \gamma_o}{\partial n_p}(P,q) + \sum_{\ell=0}^{N} \alpha_\ell \frac{\partial v_\ell}{\partial n_p}(P) v_\ell(q)\Big] \Big[\frac{\partial \bar{\gamma}_o}{\partial n_p}(P,q_1) + \sum_{\ell=0}^{N} \bar{\alpha}_\ell \frac{\partial}{\partial n_p} \bar{v}_\ell(P)\bar{v}_\ell(q_1)\Big] w(q)\bar{w}(q_1)\cdot$$

$$\cdot\, ds_q\, ds_{q_1}\, ds_p \qquad (85)$$

and we have a standard problem of minimizing a quadratic form. The necessary conditions for a minimum are the vanishing of the gradient (with respect to the coefficients) and, since the α_ℓ may be complex, the derivatives with respect to the real and imaginary parts of ∂_ℓ will vanish separately. Thus

$$\int_{\partial D}\int_{\partial D}\int_{\partial D} \frac{\partial v_m}{\partial n_p}(P) v_m(q) \Big[\frac{\partial \bar{\gamma}_o}{\partial n_p}(P,q) + \sum_{\ell=0}^{N} \bar{\alpha}_\ell \frac{\partial}{\partial n_p} \bar{v}_\ell(P)\bar{v}_\ell(q_1)\Big] \pm$$

$$\pm \frac{\partial \bar{v}_m}{\partial n_p}(P)\bar{v}_m(q_1) \Big[\frac{\partial \gamma_o}{\partial n_p}(P,q) + \sum_{\ell=0}^{N} \alpha_\ell \frac{\partial v_\ell}{\partial n_p}(P) v_\ell(q)\Big]\Big\} w(q)\bar{w}(q_1)\, ds_q\, ds_{q_1}\, ds_p = 0$$

$$m = 0, 1, 2, \ldots \qquad (86)$$

and therefore

$$\int_{\partial D}\int_{\partial D}\int_{\partial D} \frac{\partial \bar{v}_m}{\partial n_p}(P)(\bar{v}_m(q) \Big[\frac{\partial \gamma_o}{\partial n_p}(P,q) + \sum_{\ell=0}^{N} \alpha_\ell \frac{\partial v_\ell}{\partial n_p}(P) v_\ell(q)\Big] w(q)\bar{w}(q)\, ds_q\, ds_{q_1}\, ds_p =$$

$$= (\bar{v}_m, w)_{\partial D} \int_{\partial D}\int_{\partial D} \frac{\partial \bar{v}_m}{\partial n_p}(P) \Big[\frac{\partial \gamma_o}{\partial n_p}(P,q) + \sum_{\ell=0}^{N} \alpha_\ell \frac{\partial v_\ell}{\partial n_p}(P) v_\ell(q)\Big] w(q)\, ds_q\, ds_p = 0$$

$$m \geq 0\,. \qquad (87)$$

Since this is to be true for all $w \in L_2(\partial D)$ we must have

$$\int_{\partial D} \frac{\partial \bar{v}_m}{\partial n_p}(P) \Big[\frac{\partial \gamma_o}{\partial n_p}(P,q) + \sum_{\ell=0}^{N} \alpha_\ell \frac{\partial v_\ell}{\partial n_p}(P) v_\ell(q)\Big] ds_p = 0, \quad m \geq 0 \qquad (88)$$

Forming the inner product with $v_m^{\perp}(q)$ and using (81) estblishes that α_ℓ is given by (84).

It remains to show that this choice of coefficients provides a minimum. That is, if we denote by K_1^o the modified operator with coefficients as specified by (84) and by K_1 the modified operator with any other choice, we must verify that

$$||K_1^o w|| \le ||K_1 w|| \quad \text{for all } w \in L_2(\partial D).$$

Let the coefficients in the modification be denoted by

$$\alpha_\ell = \alpha_\ell^o + \varepsilon_\ell \tag{89}$$

where α_ℓ^o are defined by (84). Then

$$||K_1 w||^2_{L_2(\partial D)} = \int_{\partial D} \left| \int_{\partial D} \left\{ \frac{\partial \gamma_o}{\partial n_p}(P,q) + \sum_{\ell=0}^{N} (\alpha_\ell^o + \varepsilon_\ell) \frac{\partial v_\ell}{\partial n_p}(P) v_\ell(q) \right\} w(q)\, ds_q \right|^2 ds_P =$$

$$= ||K_1^o w||^2_{L_2(\partial D)} + \int_{\partial D}\int_{\partial D}\int_{\partial D} \sum_{\ell=0}^{N}\sum_{m=0}^{N} \varepsilon_\ell \bar{\varepsilon}_m \frac{\partial v_\ell}{\partial n_p}(P) v_\ell(q) \frac{\partial \bar{v}_m}{\partial n_p}(P) \bar{v}_m(q_1) w(q) \bar{w}(q_1) \times$$

$$\times \; ds_q \, ds_{q_1} \, ds_P =$$

$$= ||K_1^o w||^2_{L_2(\partial D)} + \sum_{\ell=0}^{N}\sum_{m=0}^{N} \varepsilon_\ell \bar{\varepsilon}_m \left(\frac{\partial v_\ell}{\partial n}, \frac{\partial v_m}{\partial n}\right)_{\partial D} (v_\ell, \bar{w})_{\partial D} (\bar{v}_m, w)_{\partial D} \tag{90}$$

where the linear terms in ε_ℓ vanish because of the choice of α_ℓ^o. Upon making the substitution

$$z_\ell := \varepsilon_\ell (v_\ell, \bar{w})_{\partial D}$$

the validity of our result is seen to depend on the positivity of the quadratic form $\displaystyle\sum_{\ell=0}^{N}\sum_{m=0}^{N} z_\ell \bar{z}_m \left(\frac{\partial v_\ell}{\partial n}, \frac{\partial v_m}{\partial n}\right)_{\partial D}$.

But, constructing an orthonormal set $\{u_\ell\}_{\ell=0}^{N}$ from $\left\{\frac{\partial v_\ell}{\partial n}\right\}_{\ell=0}^{N}$, e.g. using a Gram-Schmidt procedure and the linear independence of $\left\{\frac{\partial v_\ell}{\partial n}\right\}$

(Lemma 1), there exists a set of coefficients $\{C_{\ell i}\}$ such that

$$\frac{\partial v_\ell}{\partial n} = \sum_{i=0}^{N} C_{\ell i}\, u_i$$

for each ℓ. Then

$$\left(\frac{\partial v_\ell}{\partial n}, \frac{\partial v_m}{\partial n}\right)_{\partial D} = \sum_{i=0}^{N} \sum_{j=0}^{N} C_{\ell i}\, \bar{C}_{mj}\, (u_i, u_j)_{\partial D}$$

$$= \sum_{i=0}^{N} C_{\ell i}\, \bar{C}_{mi} = CC^* \tag{91}$$

where C is the matrix with elements $C_{\ell i}$ and C^* is the Hermitian conjugate. However CC^* is positive semidefinite [14, P69] which completes the proof.

We remark that if $\frac{\partial \gamma_o}{\partial n_p}(P,q)\Big|_{q\varepsilon\partial D} \varepsilon L_2(\partial D)$, which is true in $\mathbb{R}^2$ but not in $\mathbb{R}^3$, one could also choose the coefficients to minimize $\int_{\partial D}\int_{\partial D} \left|\frac{\partial \gamma_1}{\partial n_p}(P,q)\right|^2 ds_p\, ds_q$. However this process would lead to exactly the same coefficients specified in Theorem 5.4.

It should also be remarked that while Theorem 5.4 does provide an optimal choice for the coefficients α_ℓ, it requires the explicit construction of $v_\ell^\perp$. While this is possible with (82) it involves solving (83) for the $(N+1)^2$ coefficients C_{mj} and this certainly represents a numerical complication for large N. If the $\{v_\ell\}$ were orthogonal on ∂D i.e. when D is a sphere, then

$$v_\ell^\perp = \frac{v_\ell}{\|v_\ell\|^2_{L_2(\partial D)}} \tag{92}$$

and therefore it is proposed that even for non-spherical boundaries, a reasonable choice is provided by

$$\alpha_\ell = - \frac{\int_{\partial D}\int_{\partial D} \frac{\partial \gamma_o}{\partial n_p}(P,q)\, \frac{\partial \bar{v}_\ell}{\partial n_p}(P)\, \bar{v}_\ell(q)\, ds_p\, ds_q}{\left\|\frac{\partial v_\ell}{\partial n_p}\right\|^2_{L_2(\partial D)} \|v_\ell\|^2_{L_2(\partial D)}} \tag{93}$$

Although the consequences of this choice have not been analysed, it is expected to be useful for numerical purposes at least for boundaries which are not severe perturbations of a sphere. An even simpler approximate choice is given in section VII, motivated by further results for a spherical boundary.

VI An Example

The explicit coefficient choices found in the last section simplify considerably when the boundary is a sphere and these results are presented here. They provide the basis of a coefficient choice which, while not optimal, is convenient and hopefully useful for numerical purposes.

Approximate Dirichlet Green's Function

It is perhaps not surprising, since the modification of the Green's function is in terms of spherical wave functions, that in the case when k is real and ∂D is a sphere the coefficients in Theorem 5.1 render γ_1^D the exact Green's function, or more precisely (because of our choice of free space Green's function (3)) γ_1^D differs from the exact Green's function for the Dirichlet problem by a factor of $\frac{1}{2}$. If ∂D is a sphere of radius a then with (16) it follows that for k real,

$$a_{nm} = - \frac{\int_{\partial D} C^i_{nm}(q)\bar{C}^e_{nm}(q)\, ds_q}{\int_{\partial D} |C^e_{nm}(q)|^2 ds_q} = - \frac{j_n(ka)h_n^{(2)}(ka)}{|h_n^{(1)}(ka)|^2} = - \frac{j_n(ka)}{h_n^{(1)}(ka)} \tag{94}$$

and similarly

$$b_{nm} = -\frac{j_n(ka)}{h_n^{(1)}(ka)} \quad .$$

Then

$$\gamma_1^D(P,q) = -\frac{ik}{2\pi}\sum_{n=0}^{\infty}(2n+1)\left[j_n(kr<) - j_n\frac{(ka)h_n^{(1)}(kr<)}{h_n^{(1)}(ka)}\right]h_n^{(1)}(kr>)P_n(\cos\theta_{pq}) \tag{95}$$

where

$$\cos\theta_{pq} = \cos\theta_p\cos\theta_q + \sin\theta_p\sin\theta_q\cos(\phi_p-\phi_q) \tag{96}$$

and the relation

$$P_n(\cos\theta_{pq}) = \sum_{m=0}^{n}\varepsilon\frac{(n-m)!}{m(n+m)!}P_n^m(\cos\theta_p)P_n^m(\cos\theta_q)\cos m(\phi_p-\phi_q) \tag{97}$$

has been used. Furthermore

$$\gamma_1^D(p,q) = 0, \quad r_q = a \tag{98}$$

and, using the Wronskian (42),

$$\frac{\partial\gamma_1^D}{\partial n_q}(P,q) = \frac{1}{2\pi a^2}\sum_{n=0}^{\infty}(2n+1)\frac{h_n^{(1)}(kr_p)}{h_n^{(1)}(ka)}P_n(\cos\theta_{pq}), \; r_p > a. \tag{99}$$

If $r_p = a$ then $\frac{\partial\gamma_1^D}{\partial n_q}$ exists as a distribution in $L_2(\partial D)$ with the representation

$$\frac{\partial\gamma_1^D}{\partial n_q}(P,q) = -\frac{ik^2}{2\Pi}\sum_{n=0}^{\infty}(2n+1)\left[\frac{1}{2}j_n'(ka)h_n^{(1)}(ka) + \frac{1}{2}j_n(ka)h_n^{(1)\prime}(ka) - j_n(ka)h_n^{(1)\prime}(ka)\right] P_n(\cos\theta_{pq})$$

$$= -\frac{1}{4\pi a^2}\sum_{n=0}^{\infty}(2n+1)P_n(\cos\theta_{pq}) \tag{100}$$

With (98) we see that

$$S_1 w = 0 \quad \forall \; w \in L_2(\partial D) \tag{101}$$

therefore using the Green theorem approach, the solution of the Dirichlet problem is simply, with (52) and (101)

$$v_+ = -\frac{1}{2} D_1 f_+ = \frac{1}{4\pi} \sum_{n=0}^{\infty} (2n+1) \frac{h_n^{(1)}(kr_p)}{h_n^{(1)}(ka)} \int_0^{\pi} d\theta_q \int_0^{2\pi} d\phi_q \sin\theta_q f_+(q) P_n(\cos \theta_{pq}). \tag{102}$$

On the other hand using the layer approach we have

$$\bar{K}_1^* \bar{w}^* = -\frac{1}{4\pi} \sum_{n=0}^{\infty} (2n+1) \int_0^{\pi} d\theta_q \int_0^{2\pi} d\phi_q \sin \theta_q P_n(\cos \theta_{pq}) \bar{w}^*(q) =$$

$$= -\bar{w}^*(q) \ \forall \ \bar{w}^* \in L_2(\partial D) \tag{103}$$

because the spherical harmonics are complete on $L_2(\partial D)$. Therefore (54) becomes

$$2\bar{w}^* = f_+ \tag{104}$$

which with (53) again yields the solution (102).

Approximate Neumann Green's Function

In a manner similar to the above, when ∂D is a sphere of radius a the coefficients specified in Theorem 5.3 reduce, with (16) and orthogonality to

$$a_{nm} = b_{nm} = -\frac{j_n'(ka)}{h_n^{(1)'}(ka)} \tag{105}$$

hence the modified Green's function becomes

$$\gamma_1^N(P,q) = -\frac{ik}{2\pi} \sum_{n=0}^{\infty} (2n+1) \left[j_n(kr<) - \frac{j_n'(ka)}{h_n^{(1)'}(ka)} h_n^{(1)}(kr<) \right] h_n^{(1)}(kr>) P_n(\cos \theta_{pq}) =$$

$$= \frac{1}{2\pi k a^2} \sum_{n=0}^{\infty} (2n+1) \frac{h_n^{(1)}(kr_p)}{h_n^{(1)'}(ka)} P_n(\cos \theta_{pq}), \ r_q = a. \tag{106}$$

Moreover

$$\frac{\partial \gamma_1^N}{\partial n_q}(P,q) = 0, \quad r_p > a \tag{107}$$

while for $r_p = a$, $\frac{\partial \gamma_1^N}{\partial n_q}(P,q)$ exists as a distribution on $L_2(\partial D)$ with the representation,

$$\frac{\partial \gamma_1^N}{\partial n_q}(P,q) = -\frac{ik^2}{2\pi} \sum_{n=0}^{\infty} (2n+1) \left[\frac{1}{2} j_n'(ka) h_n^{(1)}(ka) + \frac{1}{2} j_n(ka) h_n^{(1)'}(ka) - j_n'(ka) h_n^{(1)}(ka)\right] \times P_n(\cos\theta_{pq}) =$$

$$= \frac{1}{4\pi a^2} \sum_{n=0}^{\infty} (2n+1)\, P_n(\cos\theta_{pq}). \tag{108}$$

With (107) we see that

$$D_1\, \bar{w}^* = 0,\ r_p > a\ \forall\, w^* \varepsilon L_2(\partial D). \tag{109}$$

Therefore using the Green's theorem approach the solution of the Neumann problem is with (28) and (109)

$$u_+ = \frac{1}{2} S_1 g_+ = \frac{1}{4\pi k} \sum_{n=0}^{\infty} (2n+1) \frac{h_n^{(1)}(kr_p)}{h_n^{(1)'}(ka)} \int_0^{\pi} d\theta_q \int_0^{2\pi} d\phi_q \sin\theta_q\, g_+(q) P_n(\cos\theta_{pq}). \tag{110}$$

On the other hand using the layer approach we have, with (108) and the completeness of the spherical harmonics on $L_2(\partial D)$

$$K_1 w = \frac{1}{4\pi a^2} \sum_{n=0}^{\infty} (2n+1) \int_0^{\pi} d\theta_q \int_0^{2\pi} d\phi_q \sin\theta_q w(q) P_n(\cos\theta_{pq}) = \tag{111}$$

$$= w(P)\ \forall\, w\ \varepsilon\ L_2(\partial D).$$

Therefore (30) becomes

$$2w = g_+ \tag{112}$$

which with (29) again yields the solution (110).

<u>Minimum Spectral Radius</u>

It is of some interest to compare the above results with the coefficients obtained by minimizing the spectral radius of the modified operator when ∂D is a sphere. The coefficients specified in Theorem 5.4 simplify as noted in (92) and (93). Thus with (78), (79) and (21) we have

$$\alpha_{2\ell}=a_{nm}=-\frac{\int_{\partial D}\int_{\partial D}\frac{\partial}{\partial n_p}\gamma_o(P,q)\frac{\partial}{\partial n_p}\bar{C}^e_{nm}(P)\bar{C}^e_{nm}(q)ds_q ds_p}{\left\|\frac{\partial C^e_{nm}}{\partial n}\right\|^2_{L_2(\partial D)}\left\|C^e_{nm}\right\|^2_{L_2(\partial D)}}$$

$$=\frac{\int_{\partial D}\int_{\partial D}\sum_{r=0}^{\infty}\sum_{\mu=0}^{r}\frac{1}{2}\left(C^e_{r\mu}(q)\frac{\partial}{\partial n_p}C^i_{r\mu}(P)+C^i_{r\mu}(q)\frac{\partial}{\partial n_p}C^e_{r\mu}(P)\right)\frac{\partial}{\partial n_p}\bar{C}^e_{nm}(P)\bar{C}^e_{nm}(q)ds_p ds_q}{\left\|\frac{\partial C^e_{nm}}{\partial n}\right\|^2_{L_2(\partial D)}\left\|C^e_{nm}\right\|^2_{L_2(\partial D)}}$$

$$=-\frac{\frac{1}{2}\{h_n^{(1)}(ka)j_n'(ka)+j_n(ka)h_n^{(1)'}(ka)\}h_n^{(2)'}(ka)h_n^{(2)}(ka)}{h_n^{(1)'}(ka)h_n^{(2)'}(ka)h_n^{(1)}(ka)h_n^{(2)}(ka)}$$

$$=-\frac{1}{2}\left(\frac{j_n'(ka)}{h_n^{(1)'}(ka)}+\frac{j_n(ka)}{h_n^{(1)}(ka)}\right) \tag{113}$$

A similar analysis shows that in this case

$$\alpha_{2\ell+1}=b_{nm}=a_{nm}. \tag{114}$$

With a_{nm} and b_{nm} so defined the modified Green's function becomes, using (95) and (106)

$$\gamma_1(P,q)=$$

$$=-\frac{ik}{2\pi}\sum_{n=0}^{\infty}(2n+1)\left\{j_n(kr_<)-\frac{1}{2}\left(\frac{j_n'(ka)}{h_n^{(1)'}(ka)}+\frac{j_n(ka)}{h_n^{(1)}(ka)}\right)h_n^{(1)}(ka)\right\}h_n^{(1)}(kr_p)\times$$

$$\times\ P_n(\cos\theta_{pq})$$

$$=\frac{1}{2}(\gamma^D(P,q)+\gamma^N(P,q)) \tag{115}$$

Hence with (98) (99), (106) and (107) we see that

$$\gamma_1(P,q) = \frac{1}{2}\gamma_1^N(P,q), \quad r_q = a \tag{116}$$

$$\frac{\partial\gamma_1(P,q)}{\partial n_q} = \frac{1}{2}\ \frac{\partial}{\partial n_q}\ \gamma_1^D(P,q), \quad r_p > a \tag{117}$$

Moreover with (100) and (108) we have

$$\frac{\partial\gamma_1(P,q)}{\partial n_q} = 0, \quad r_p = a. \tag{118}$$

But this last result implies that

$$K_1 w = \bar{K}_1^* w = 0 \ \forall\ w \in L_2(\partial D) \tag{119}$$

hence the spectral radius of the modified operator is zero. Observe that with (103) and (111) the spectral radii of the modified operators generated by γ_1^D and γ_1^N are both equal to one.

As a consequence of (119) we have

$$D_1(S_1 g_+) = -S_1 g_+, \quad P \in D_+ \ \forall\ g_+ \in L_2(\partial D) \tag{120}$$

and

$$S_1\left(\frac{\partial}{\partial n} D_1 f_+\right) = D_1 f_+, \ P \in D_+ \ \forall\ f_+ \in L_2(\partial D) \tag{121}$$

Equation (120) follows from the uniqueness of solutions of the exterior Dirichlet problem and observing that with (10) and (119)

$$\lim_{P\to p^+} \left[D_1(S_1 g_+) + S_1 g_+\right] = 0 \tag{122}$$

whereas (121) follows from uniqueness for the exterior Neumann problem and observing that with (9) and (119)

$$\frac{\partial}{\partial n_+}\left[S_1\left(\frac{\partial}{\partial n} D_1 f_+\right) - D_1 f_+\right] = 0 \tag{123}$$

To solve the Dirichlet problem with this modified operator we have with (51) and (119) and (52)

$$v_+ = \frac{1}{2} S_1 w - \frac{1}{2} D_1 f_+ = -\frac{1}{2} S_1\left(\frac{\partial}{\partial n} D_1 f_+\right) - \frac{1}{2} D_1 f_+, \quad P \in D_+ \tag{124}$$

but with (121) and (117) this becomes

$$v_+ = - D_1 f_+ = - \frac{1}{2} \int_{\partial D} \frac{\partial}{\partial n_q} \gamma_1^D (P,q) f_+(q) ds_q \tag{125}$$

in agreement with (102). Similarly using the layer ansatz, (54) and (119) imply that

$$\bar{w}^* = f_+$$

and then (53) yields the same solution (125).

To solve the Neumann problem, (26) and (119) yields

$$\bar{w}^* = S_1 g_+ \tag{126}$$

and with (28)

$$u_+ = \frac{1}{2} S_1 g_+ - \frac{1}{2} D_1 (S_1 g_+). \tag{127}$$

Using (120 and (116) this becomes

$$u_+ = S_1 g_+ = \frac{1}{2} \int_{\partial D} \gamma_1^N (P,q) g_+(q) ds_q \tag{128}$$

in agreement with (110). With the layer approach (30) and (119) imply

$$w = g_+ \tag{129}$$

which together with (29) again yields the solution (128).

The remarkable result (119) shows that the spectral radius of the modified operator is zero in this sphere example when none of the coefficients in the modification vanish. For applications one would like to know the effect on the spectral radius of the modified operator if only a finite number of coefficients are chosen optimally while the remainder are taken to be zero i.e. modifications of the form

$\sum_{n=0}^{N} \sum_{m=0}^{M} a_{nm} C_{nm}^e(P) C_n^e(q) + b_{nm} S_{nm}^e(P) S_{nm}^e(q)$. In this case the optimal coefficients are again given by (113) and (114). However, instead of (118) we find that for $r_p = a$,

$$\frac{\partial \gamma_1(P,q)}{\partial n_q} = -\frac{ik^2}{4\pi}\sum_{n=N+1}^{\infty}(2n+1)\{j_n'(ka)h_n^{(1)}(ka)+j_n(ka)h_n^{(1)\prime}(ka)\}P_n(\cos\theta_{pq}) \tag{130}$$

and $K_1 w = \int_{\partial D}\frac{\partial\gamma_1(P,q)}{\partial n_q}\, w(q)\, ds_q$

so that, using the orthogonality of the spherical harmonics,

$$||K_1 w||^2_{L_2(\partial D)} = \int_{\partial D}(K_1 w)^2 ds = \frac{k^4 a^2}{4\pi}\sum_{n=N+1}^{\infty}(2n+1)|j_n'(ka)h_n^{(1)}(ka)+j_n(ka)h_n^{(1)\prime}(ka)|^2 \cdot$$

$$\cdot\int_{\partial D}\int_{\partial D}P_n(\cos(\theta_{qq_1})w(q)\bar{w}(q_1)ds_q ds_{q_1}\ . \tag{131}$$

But $w(q)\ \varepsilon\ L_2(\ D) \Rightarrow w(q) = \sum_{n=0}^{\infty} Y_n(\theta_q,\phi_q)$ where Y_n are general spherical harmonics and

$$\int_{\partial D}\int_{\partial D}P_n(\cos(\theta_{qq_1})w(q)\bar{w}(q)ds_q ds_{q_1} =$$

$$= \frac{4\pi a^2}{2n+1}\int_{\partial D}|Y_n(\theta_q,\phi_{q_1})|^2 ds_{q_1} \leq \frac{4\pi}{2n+1}||w||^2_{L_2}(\partial D)\ . \tag{132}$$

Therefore

$$||K_1 w||^2_{L_2(\partial D)} \leq (ka)^4 ||w||^2_{L_2(\partial D)}\sum_{n=N+1}^{\infty}|j_n'(ka)h_n^{(1)}(ka)+j_n(ka)h_n^{(1)\prime}(ka)|^2\ . \tag{133}$$

Using very crude estimates for the spherical Bessel and Hankel functions it may be shown that there exists a constant C, independent of ka and N, such that

$$|j_n'(ka)h_n^{(1)}(ka)+j_n(ka)h_n^{(1)\prime}(ka)|^2 < \frac{Ce^{\frac{(ka)^2}{2}}}{(ka)^4(2n+1)^2} \tag{134}$$

hence

$$||K_1 w||^2_{L_2(\partial D)} \leq ||w||^2_{L_2(\partial D)} Ce^{\frac{(ka)^2}{2}} \sum_{n=N+1}^{\infty} \frac{1}{(2n+1)^2} < ||w||^2_{L_2(\partial D)} \frac{Ce^{\frac{(ka)^2}{2}}}{N+1} \tag{135}$$

and

$$||K_1|| < \frac{Ce^{\frac{(ka)^2}{2}}}{N+1} . \tag{136}$$

Thus for any value of ka it is clear that $||K_1|| < 1$ for N large enough, and therefore the spectral radius of the modified operator can be made less than one with only a finite number of nonzero coefficients in the modification.

VII Concluding Remarks

In this paper we have shown how the modified Green's function approach of Jones and Ursell can be extended to both Dirichlet and Neumann problems for the Helmholtz equation in three dimensions giving rise to an integral equation of the second kind that is uniquely solvable for all real values of k. In addition we have shown how to choose the coefficients in the modification optimally, either to best approximate the Dirichlet or Neumann Green's function or to minimize the norm of the modified operator. The optimal results were exhibited explicitly for the sphere where it was also shown that only a finite number of coefficients need be chosen different from zero to force the spectral radius of the integral operator to be less than one for any finite value of ka.

The coefficients for which the modified Green's function best approximates the exact Green's function are given explicitly (Theorems 5.1 and 5.3). However the coefficients which minimize the spectral radius (Theorem 5.4) require the construction of a dual basis for $\{C^e_{nm}, S^e_{nm}\}$ on ∂D. For nonspherical ∂D and large values of N this may be a considerable numerical complication and one possible simplification has been proposed, (93). The explicit results for the sphere allow still another even simpler coefficient choice. Comparing the explicit results for the coefficients which give the best approximation to the Dirichlet and Neumann Green's functions when ∂D is a sphere, equation (94) and (105) with the coefficients which minimize the norm of the modified operator (113), we find that these latter coefficients may be written as

$$a_{nm} = -\frac{1}{2}\left\{\frac{(C^{i}_{nm}, C^{e}_{nm})_{\partial D}}{||C^{e}_{nm}||^{2}_{L_2(\partial D)}} + \frac{\left(\frac{\partial C^{i}_{nm}}{\partial n}, \frac{\partial C^{e}_{nm}}{\partial n}\right)_{\partial D}}{||\frac{\partial}{\partial n} C^{e}_{nm}||^{2}_{L_2(\partial D)}}\right\} \tag{137}$$

and

$$b_{nm} = -\frac{1}{2}\left\{\frac{(S^{i}_{nm}, S^{e}_{nm})_{\partial D}}{||S^{e}_{nm}||^{2}_{L_2(\partial D)}} + \frac{\left(\frac{\partial}{\partial n} S^{i}_{nm}, \frac{\partial}{\partial n} S^{e}_{nm}\right)_{\partial D}}{||\frac{\partial}{\partial n} S^{e}_{nm}||^{2}_{L_2(\partial D)}}\right\} \tag{138}$$

While the coefficients thus defined minimize the norm of K_1 only when ∂D is a sphere, it is proposed that this choice be used for nonspherical surfaces as well. It is easy to demonstrate that conditions (33) and (34) which ensure that K_1 has no characteristic values is fulfilled since

$$\begin{aligned}|2a_{nm}+1| &= \left|\frac{1}{2} - \frac{(C^{i}_{nm}, C^{e}_{nm})_{\partial D}}{||C^{e}_{nm}||^{2}_{L_2(\partial D)}} + \frac{1}{2} - \frac{\left(\frac{\partial C^{i}_{nm}}{\partial n}, \frac{\partial C^{e}_{nm}}{\partial n}\right)_{\partial D}}{||\frac{\partial}{\partial n} C^{e}_{nm}||^{2}_{L_2(\partial D)}}\right| \\ &= \left| - \frac{(\bar{C}^{e}_{nm}, C^{e}_{nm})_{\partial D}}{2||C^{e}_{nm}||^{2}_{L_2(\partial D)}} - \frac{\left(\frac{\partial \bar{C}^{e}_{nm}}{\partial n}, \frac{\partial C^{e}_{nm}}{\partial n}\right)_{\partial D}}{2||\frac{\partial}{\partial n} C^{e}_{nm}||^{2}_{L_2(\partial D)}}\right| \\ &\le \frac{|(\bar{C}^{e}_{nm}, C^{e}_{nm})_{\partial D}|}{2||C^{e}_{nm}||^{2}_{L_2(\partial D)}} + \frac{1}{2} < 1\end{aligned} \tag{139}$$

where the last inequality follows from Theorem 5.2. A similar analysis shows that $|2b_{nm}+1| < 1$. It is reasonable to expect that with the coefficients defined by (137) and (138) the spectral radius of the modified operator, which is zero when ∂D is a sphere, will remain less than one for a class of nonspherical surfaces and furthermore if only a finite number of coefficients are taken to nonzero, defined by (137) and (138), the spectral radius will be less one in an interval in k which increases with the number of nonzero coefficients. A characterization of this class of surfaces and relationship between the geometry, values of k for which the spectral radius is less than one, and the number of nonzero coefficients constitute a class of problems that remain to be solved.

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